New Classes of Multivariate Survival Functions

Xiao Qin[12] Richard L. Smith[2] Ruoen Ren[1]

[1]School of Economics and Management
Beihang University
Beijing, China

[2]Department of Statistics and Operations Research
University of North Carolina
Chapel Hill, NC, USA

SAMSI workshop, May 21, 2008

Outline of Topics

Multivariate Extreme Value model

Define i.i.d p-dimensional random vectors: $\mathbf{X}_k = (X_{1,k}, \cdots, X_{p,k})'$, $k = 1, \cdots, n$ with marginal distribution $\Pr(X_{i,k} \leq x_i) = F_{X_i}(x_i)$, $i = 1, \cdots, p$.

The ith component maximum: $M_{i,n} = \max(X_{i,1}, \cdots, X_{i,n})$

Choose normalising constants $\mathbf{a}_n = (a_{1,n}, \cdots, a_{p,n})'$ and $\mathbf{b}_n = (b_{1,n}, \cdots, b_{p,n})'$ such that,

$$\lim_{n\to\infty} \Pr\left\{\frac{M_{i,n} - b_{i,n}}{a_{i,n}} \leq x_i\right\} = G(x_1, \cdots, x_d)$$

Denoting by vectors, we get:

$$\lim_{n\to\infty} \Pr\left\{\frac{\mathbf{M}_n - \mathbf{b}_n}{\mathbf{a}_n}\right\} = G(\mathbf{x})$$

Multivariate Extreme Value model

It is easily checked that all the marginal distributions of G are Generalised Extreme Value distribution (GEV), which belongs to one of the following three families (Fisher-Tippett theorem).

$$F_{X_i}(x_i) = G(\infty, \cdots, \infty, x_i, \infty, \cdots, \infty)$$
$$= \begin{cases} \mathit{Gumbel}: & \exp\left\{-\exp(-x_i)\right\} \\ \mathit{Fréchet}: & \exp\left\{-x_i^{-\alpha}\right\} \\ \mathit{Weibull}: & \exp\left\{-(-x_i)^{\alpha}\right\} \end{cases}$$

A common practice is to choose the Fréchet marginals, and usually the unit Fréchet case when shape parameter $= 1/\alpha = 1$.
This is WOLG, as any marginal distributions can be changed into a unit Fréchet by a probability integral transformation. Details can be found in Smith (2003).

Multivariate Extreme Value model

Pickands' representation: For $\mathbf{x}$ with unit Fréchet marginals, $\mathbf{x}$ follows a multivariate extreme value distribution iff its joint distribution function can be represented as:

$$G(\mathbf{x}) = \exp\left\{-V(\mathbf{x})\right\} \tag{1}$$

Here, $V(x)$ is the exponent measure as in de Haan and Resnick (1977) and can be expressed in terms of an angular measure H (density h) as:

$$V(\mathbf{x}) = \int_{S_p} \max_{1 \le i \le p} \left(\frac{w_i}{x_i}\right) dH(\mathbf{w}) \tag{2}$$

Multivariate Extreme Value model

Where, H is a positive finite measure on the $(p-1)$-simplex,

$$S_p = \left\{ (w_1, ..., w_p) : \sum_{j=1}^{p} w_j = 1, w_j \geq 0, j = 1, \cdots, p \right\}$$

satisfying the following constraint.
MEV constraint: Suppose $x_i = u$, $x_j = \infty$, $j \neq i$, $i = 1, \cdots, p$, then, $V(\infty, \cdots, \infty, u, \infty, \cdots, \infty) = u^{-1}$. Thus,

$$1 = \int_{S_p} w_i \, dH(\mathbf{w}) = V(\infty, \cdots, \infty, 1, \infty, \cdots, \infty) \tag{3}$$

wherever the *1* is.

Multivariate Extreme Value model

General case:

$$V_\eta(\mathbf{x}) = \int_{S_p} \eta \left\{ \max_{1 \le i \le p} \left(\frac{w_i}{x_i} \right) \right\}^{\frac{1}{\eta}} dH_\eta(\mathbf{w}) \tag{4}$$

MEV constraint (general):

$$1 = \int_{S_p} \eta w_i^{\frac{1}{\eta}} \, dH_\eta(\mathbf{w}) = V_\eta(\infty, \cdots, \infty, 1, \infty, \cdots, \infty) \tag{5}$$

wherever the 1 is.
In this case, $G_\eta(\mathbf{x})$ has Fréchet marginals with shape parameter η and scale parameter *1*.

Limitation of the Multivariate Extreme Value model

As stated in Ledford and Tawn (1996), all Bivariate Extreme Value models with Fréchet marginals have:

$$\lim_{r\to\infty} \Pr(X_1 > r, X_2 > r) = \begin{cases} r^{-1} & \textit{asymptotic dependence} \\ r^{-2} & \textit{exact independence} \end{cases}$$

However, strictly speaking, there should also be negative association and positive association under asymptotic independence.

Ramos-Ledford-Tawn model

Ledford-Tawn model: Ledford and Tawn (1997)
For $(Z_{1k}, Z_{2k}), k = 1, \cdots, n$ with unit Fréchet marginals but unknown dependence structure, the joint survival functions $\bar{F}_{Z_1 Z_2}$ has the following asymptotic form:

$$\bar{F}_{Z_1 Z_2}(z_1, z_2) = L(z_1, z_2) z_1^{-c_1} z_2^{-c_2}$$

where, $c_1 + c_2 = \frac{1}{\eta}$ (η is called the **Coefficient of tail dependence**), $L(z_1, z_2)$ is a bivariate slowly varying function (BSV) with limiting function g, i.e., $\lim_{t \to \infty} \frac{L(tz_1, tz_2)}{L(t,t)} = g(z_1, z_2)$ and $g(cz_1, cz_2) = g(z_1, z_2)$, $\forall c > 0, z_1, z_2 > 0$.

Ramos-Ledford-Tawn model

Ramos and Ledford (2007) reformulated the Ledford-Tawn model:

$$\bar{F}_{Z_1 Z_2}(z_1, z_2) = \tilde{L}(z_1, z_2)(z_1 z_2)^{-\frac{1}{2\eta}}$$

where,

$$\tilde{L}(z_1, z_2) = L(z_1, z_2)(\frac{z_1}{z_2})^{\kappa/2}, \kappa = c_2 - c_1$$

Consider the limiting joint survival distribution of the conditional variables $(X_1, X_2) = (Z_1/u, Z_2/u)|(Z_1 > u, Z_2 > u)$, thus, $X_1, X_2 \geq 1$. Then,

$$\bar{F}_{X_1 X_2}(x_1, x_2) = \lim_{u \to \infty} \frac{\Pr(Z_1 > ux_1, Z_2 > ux_2)}{\Pr(Z_1 > u, z_2 > u)}$$
$$= \lim_{u \to \infty} \frac{\tilde{L}(ux_1, ux_2)}{\tilde{L}(u, u)} \frac{1}{(x_1 x_2)^{1/2\eta}}$$
$$= \frac{g(x_1, x_2)}{(x_1 x_2)^{1/2\eta}} = \frac{g_*(\frac{x_1}{x_1 + x_2})}{(x_1 x_2)^{1/2\eta}}$$

Ramos-Ledford-Tawn model

Ramos-Ledford spetral model:
By taking a polar-coordinate transformation, the following relationship of the limiting survivor in terms of an angular measure H^*_η is obtained:

$$\bar{F}_{X_1X_2}(x_1x_2) = \int_0^1 \eta \left\{ \min(\frac{w}{x_1}, \frac{1-w}{x_2}) \right\}^{\frac{1}{\eta}} dH^*_\eta(w)$$

Ramos-Ledford Constraint: Suppose $x_1 = x_1 = u$, then, $\bar{F}_{X_1X_2}(u,u) = u^{-\frac{1}{\eta}}$. Thus, the following normalising condition should hold:

$$1 = \int_0^1 \eta \left\{ \min(\frac{w}{x_1}, \frac{1-w}{x_2}) \right\}^{\frac{1}{\eta}} dH^*_\eta(w) = \bar{F}_{X_1X_2}(1,1)$$

Ramos-Ledford-Tawn model

Multivariate version:
Define i.i.d p-dimensional random vectors $\mathbf{Z}_k = (Z_{1,k}, \cdots, Z_{p,k})'$, $k = 1, \cdots, n$, with unit Fréchet marginals. By assuming multivariate regular variation, we have

$$\bar{F}_{Z_1 \cdots Z_p}(z_1, \cdots, z_p) = L(z_1, \cdots, z_p) z_1^{-c_1} \cdots z_p^{-c_p} \qquad (6)$$

where, $z_1, \cdots, z_p > 0$, $c_1 + \cdots + c_p = \frac{1}{\eta}$, L is a multivariate regularly varying function (MRV) satisfying:

$$\lim_{t \to \infty} \frac{L(tz_1, \cdots, tz_p)}{L(t, \cdots, t)} = g(z_1, \cdots, z_p)$$

and $g(cz_1, \cdots, cz_p) = g(z_1, \cdots, z_p)$, $\forall c > 0, z_1, \cdots, z_p > 0$.

Ramos-Ledford-Tawn model

Rewrite the Eq. (6), we get,

$$\bar{F}_{Z_1 \cdots Z_p}(z_1, \cdots, z_p) = \tilde{L}(z_1, \cdots, z_p)(z_1 \cdots z_p)^{-\frac{1}{p\eta}} \quad (7)$$

where, $\tilde{L} = L \cdot \left(\frac{z_1}{z_2}\right)^{\kappa_{12}/p} \left(\frac{z_2}{z_3}\right)^{\kappa_{23}/p} \cdots \left(\frac{z_1}{z_p}\right)^{\kappa_{1p}/p}$,
$\kappa_{12} = c_2 - c_1, \kappa_{23} = c_3 - c_2, \cdots, \kappa_{1p} = c_p - c_1$.
Define conditional variables:
$(X_1, \cdots, X_p) = (Z_1/u, \cdots, Z_p/u)|(Z_1 > u, \cdots, Z_p > u)$, thus, $X_1, \cdots, X_p \geq 1$. Then,

$$\bar{F}_{X_1 \cdots X_p}(x_1, \cdots, x_p) = \lim_{u \to \infty} \frac{\Pr(Z_1 > ux_1, \cdots, Z_p > ux_p)}{\Pr(Z_1 > u, \cdots, Z_p > u)}$$
$$= \lim_{u \to \infty} \frac{\tilde{L}(ux_1, \cdots, ux_p)}{\tilde{L}(u, \cdots, u)} \frac{1}{(x_1 \cdots x_p)^{1/p\eta}} \quad (8)$$
$$= \frac{g(x_1, \cdots, x_p)}{(x_1 \cdots x_p)^{1/p\eta}} = \frac{g_*\left(\frac{x_1}{\sum_{i=1}^p x_i}, \cdots, \frac{x_{p-1}}{\sum_{i=1}^p x_i}\right)}{(x_1 \cdots x_p)^{1/p\eta}}$$

Ramos-Ledford-Tawn model

By taking a polar-coordinate transformation again, we have:

$$\bar{F}_{X_1 \cdots X_p}(x_1, \cdots, x_p) = \int_{S_p} \eta \left\{ \min_{1 \le i \le p} \left(\frac{w_i}{x_i} \right) \right\}^{\frac{1}{\eta}} dH_\eta^*(\mathbf{w}) \qquad (9)$$

Ramos-Ledford Constraint (general): Suppose $x_i = u, i = 1, \cdots, p$, then, $\bar{F}_{X_1 \cdots X_p}(u, \cdots, u) = u^{-\frac{1}{\eta}}$. Thus, the following normalising condition should hold:

$$1 = \int_{S_p} \eta \left\{ \min_{1 \le i \le p} w_i \right\}^{\frac{1}{\eta}} dH_\eta^*(\mathbf{w}) = \bar{F}_{X_1 \cdots X_p}(1, \cdots, 1) \qquad (10)$$

From h_0 to h_η

Theorem 1: Given any positive function h_0 with finite $\frac{1}{\eta}$ moments on S_p, then,

$$h_\eta(\mathbf{w}) = (\mathbf{m}^\eta \mathbf{w})^{-(p+\frac{1}{\eta})} \prod_{i=1}^{p} m_i^\eta h_0\left(\frac{m_1^\eta w_1}{\mathbf{m}^\eta \mathbf{w}}, \cdots, \frac{m_p^\eta w_p}{\mathbf{m}^\eta \mathbf{w}}\right) \quad (11)$$

where,

$$m_i = \int_{S_p} \eta u_i^{\frac{1}{\eta}} h_0(\mathbf{u})\, d\mathbf{u}, i = 1, \cdots, p \quad (12)$$

is the density of a valid measure function H_η, satisfying the MEV constraint (general), i.e., Eq. (5).
The special case when $\eta = 1$ reduces to the result in Coles and Tawn (1991).

From h_η to h^*_η

Theorem 2: Given any function h_η satisfying the MEV constraint (general), i.e., Eq. (5), then,

$$h^*_\eta(\mathbf{w}) = \frac{h_\eta(\mathbf{w})}{\delta} \tag{13}$$

where,

$$\delta = p + (-1)^3 \sum V_\eta(\infty, \cdots, \infty, 1, \infty, \cdots, \infty, 1, \infty, \cdots, \infty) + \cdots + (-1)^{p+1} V_\eta(1, \cdots, 1) \tag{14}$$

is a valid density function of a measure function H^*_η, satisfying the Ramos-Ledford Constraint (general), i.e., Eq. (10).
The special case when $d = 2$ and $\eta = 1$ is given in Ramos and Ledford (2007), where, $h^*(w) = h(w)/(2 - V(1, 1))$.

From h_η to h^*_η

Sketch of the proof: It is straightforward to check that,

$$\int_{S_p} \eta \left\{ \min_{1 \le i \le p} w_i \right\}^{\frac{1}{\eta}} h^*_\eta(\mathbf{w})\, d\mathbf{w}$$

$$= \int_{S_p} \eta \left\{ \min_{1 \le i \le p} w_i \right\}^{\frac{1}{\eta}} \frac{h_\eta(\mathbf{w})}{\int_{S_p} \eta \left\{ \min_{1 \le i \le p} w_i \right\}^{\frac{1}{\eta}} h_\eta(\mathbf{w})\, d\mathbf{w}}\, d\mathbf{w} = 1$$

From h_η to h_η^*

According to the inclusion-exclusion principle,

$$\sigma = \int_{S_p} \eta \left\{ \min_{1\leq i \leq p} w_i \right\}^{\frac{1}{\eta}} h_\eta(\mathbf{w})\, d\mathbf{w}$$

$$= (-1)^2 \sum_{i=1}^{p} \int_{S_p} \eta w_i^{\frac{1}{\eta}} h_\eta(\mathbf{w})\, d\mathbf{w}$$

$$+(-1)^3 \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \int_{S_p} \eta \left\{ \max_{\substack{i\neq j, 1\leq i\leq p, \\ i+1\leq j\leq p}} (w_i, w_j) \right\}^{\frac{1}{\eta}} h_\eta(\mathbf{w})\, d\mathbf{w}$$

$$+ \cdots + (-1)^{p+1} \int_{S_p} \eta \left\{ \max_{1\leq i\leq p} w_i \right\}^{\frac{1}{\eta}} h_\eta(\mathbf{w})\, d\mathbf{w}$$

General survival functions

Following the definition of $\bar{F}_{X_1 \cdots X_p}(x_1, \cdots, x_p)$ in terms of angular measures, we get,

$$\begin{aligned}
&\bar{F}_{X_1 \cdots X_p}(x_1, \cdots, x_p) \\
&= \int_{S_p} \eta \left\{ \min_{1 \le i \le p} (\frac{w_i}{x_i}) \right\}^{\frac{1}{\eta}} h^*_\eta(\mathbf{w})\, d\mathbf{w} \\
&= \delta^{-1} \int_{S_p} \eta \left\{ \min_{1 \le i \le p} (\frac{w_i}{x_i}) \right\}^{\frac{1}{\eta}} h_\eta(\mathbf{w})\, d\mathbf{w} \qquad (15) \\
&= \eta \delta^{-1} \int_{S_p} \left\{ \min_{1 \le i \le p} (\frac{w_i}{x_i}) \right\}^{\frac{1}{\eta}} (\mathbf{m}^\eta \mathbf{w})^{-(p + \frac{1}{\eta})} \\
&\quad \prod_{i=1}^p m_i^\eta h_0(\frac{m_1^\eta w_1}{\mathbf{m}^\eta \mathbf{w}}, \cdots, \frac{m_p^\eta w_p}{\mathbf{m}^\eta \mathbf{w}})\, d\mathbf{w}
\end{aligned}$$

General survival functions

As $\bar{F}_{Z_1 \cdots Z_p}(z_1, \cdots, z_p) = \lambda \bar{F}_{X_1 \cdots X_p}(\frac{z_1}{u}, \cdots, \frac{z_p}{u})$, where $\lambda = \Pr(Z_1 > u, \cdots, Z_p > u)$.

$$\bar{F}_{Z_1 \cdots Z_p}(z_1, \cdots, z_p) = \eta \lambda u^{\frac{1}{\eta}} \delta^{-1} \int_{S_p} \left\{ \min_{1 \le i \le p} \left(\frac{w_i}{z_i} \right) \right\}^{\frac{1}{\eta}} (\mathbf{m}^{\eta}\mathbf{w})^{-(p+\frac{1}{\eta})} \prod_{i=1}^{p} m_i^{\eta} h_0\left(\frac{m_1^{\eta} w_1}{\mathbf{m}^{\eta}\mathbf{w}}, \cdots, \frac{m_p^{\eta} w_p}{\mathbf{m}^{\eta}\mathbf{w}}\right) d\mathbf{w} \tag{16}$$

where, as defined in Theorem 1 and 2,

$$m_i = \int_{S_p} \eta u_i^{\frac{1}{\eta}} h_0(\mathbf{u})\, d\mathbf{u},\ i = 1, \cdots, p$$

$$\delta = p + (-1)^3 \sum V_{\eta}(\infty, \cdots, \infty, 1, \infty, \cdots, \infty, 1, \infty, \cdots, \infty) + \cdots + (-1)^{p+1} V_{\eta}(1, \cdots, 1)$$

Validility check

We check the validility of $\bar{F}_{Z_1Z_2}$ as a survival function, i.e., for $x_1 \leq x_2, y_1 \leq y_2$,

$$\begin{aligned}
&\bar{F}_{Z_1Z_2}(x_2,y_2) - \bar{F}_{Z_1Z_2}(x_1,y_2) - \bar{F}_{Z_1Z_2}(x_2,y_1) + \bar{F}_{Z_1Z_2}(x_1,y_1) \\
&= \lambda u^{\frac{1}{\eta}} \delta^{-1} \left\{ \left[x_1^{-\frac{1}{\eta}} \int_{\frac{x_1}{x_1+y_2}}^{\frac{x_1}{x_1+y_1}} \eta w^{\frac{1}{\eta}} h_\eta(w)\, dw - x_2^{-\frac{1}{\eta}} \int_{\frac{x_2}{x_2+y_2}}^{\frac{x_2}{x_2+y_1}} \eta w^{\frac{1}{\eta}} h_\eta(w)\, dw \right] \right. \\
&\qquad + \left[y_1^{-\frac{1}{\eta}} \int_{\frac{x_1}{x_1+y_1}}^{\frac{x_2}{x_2+y_1}} \eta (1-w)^{\frac{1}{\eta}} h_\eta(w)\, dw \right. \\
&\qquad\qquad \left.\left. - y_2^{-\frac{1}{\eta}} \int_{\frac{x_1}{x_1+y_2}}^{\frac{x_2}{x_2+y_2}} \eta (1-w)^{\frac{1}{\eta}} h_\eta(w)\, dw \right] \right\} \\
&\geq 0
\end{aligned}$$

A theoretical example

Take the Dirichlet distribution for example,

$$h_0(w) = \left\{ \prod_{i=1}^{p} \Gamma(\alpha_i) \right\}^{-1} \Gamma(\alpha \cdot \mathbf{1}) \prod_{i=1}^{p} w_i^{\alpha_i - 1}$$

According to the definition of m_i, we have,

$$m_i = \eta \frac{\Gamma(\alpha \cdot \mathbf{1}) \Gamma(\alpha_i + \frac{1}{\eta})}{\Gamma(\alpha_i) \Gamma(\alpha \cdot \mathbf{1} + \frac{1}{\eta})}, i = 1, \cdots, p$$

A theoretical example

By Theorem 1, we get,

$$h_\eta(\mathbf{w}) = \eta^{-1} \prod_{i=1}^{p} \frac{\Gamma^\eta(\alpha_i + \frac{1}{\eta})}{\Gamma^{\eta+1}(\alpha_i)} \frac{\Gamma(\alpha\cdot\mathbf{1} + \frac{1}{\eta})}{\left\{\sum_{i=1}^{p} \frac{\Gamma^\eta(\alpha_i+\frac{1}{\eta})}{\Gamma^\eta(\alpha_i)} w_i\right\}^{p+\frac{1}{\eta}}}$$

$$\prod_{i=1}^{p} \left\{ \frac{\frac{\Gamma^\eta(\alpha_i+\frac{1}{\eta})}{\Gamma^\eta(\alpha_i)} w_i}{\sum_{i=1}^{p} \frac{\Gamma^\eta(\alpha_i+\frac{1}{\eta})}{\Gamma^\eta(\alpha_i)} w_i} \right\}^{\alpha_i - 1} \quad (17)$$

$$= \eta^{-1} \prod_{i=1}^{p} \frac{\gamma_i}{\Gamma(\alpha_i)} \frac{\Gamma(\alpha\cdot\mathbf{1} + \frac{1}{\eta})}{(\gamma\cdot\mathbf{w})^{p+\frac{1}{\eta}}} \prod_{i=1}^{p} \left\{ \frac{\gamma_i w_i}{\gamma\cdot\mathbf{w}} \right\}^{\alpha_i - 1}$$

denoting

$$\gamma = (\gamma_1, \cdots, \gamma_p)' = \left(\frac{\Gamma^\eta(\alpha_1 + \frac{1}{\eta})}{\Gamma^\eta(\alpha_1)}, \cdots, \frac{\Gamma^\eta(\alpha_p + \frac{1}{\eta})}{\Gamma^\eta(\alpha_p)} \right)' \quad (18)$$

A theoretical example

Following Theorem 2, we get,

$$h_\eta^*(\mathbf{w}) = \eta^{-1}\delta^{-1}_{\alpha_1,\cdots,\alpha_p,\eta}\prod_{i=1}^{p}\frac{\gamma_i}{\Gamma(\alpha_i)}\frac{\Gamma(\alpha\cdot\mathbf{1}+\frac{1}{\eta})}{(\gamma\cdot\mathbf{w})^{p+\frac{1}{\eta}}}\prod_{i=1}^{p}\left\{\frac{\gamma_i w_i}{\gamma\cdot\mathbf{w}}\right\}^{\alpha_i-1} \tag{19}$$

where,

$$\delta^{-1}_{\alpha_1,\cdots,\alpha_p,\eta} = p + (-1)^3\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\left[\int_{S_p}\left\{\max_{\substack{i\neq j,1\leq i\leq p,\\ i+1\leq j\leq p}}(w_i,w_j)\right\}^{\frac{1}{\eta}}\right.$$

$$\left.\prod_{i=1}^{p}\frac{\gamma_i}{\Gamma(\alpha_i)}\frac{\Gamma(\alpha\cdot\mathbf{1}+\frac{1}{\eta})}{(\gamma\cdot\mathbf{w})^{p+\frac{1}{\eta}}}\prod_{i=1}^{p}\left\{\frac{\gamma_i w_i}{\gamma\cdot\mathbf{w}}\right\}^{\alpha_i-1}d\mathbf{w}\right]+\cdots+$$

$$(-1)^{p+1}\int_{S_p}\left\{\max_{1\leq i\leq p} w_j\right\}^{\frac{1}{\eta}}\prod_{i=1}^{p}\frac{\gamma_i}{\Gamma(\alpha_i)}\frac{\Gamma(\alpha\cdot\mathbf{1}+\frac{1}{\eta})}{(\gamma\cdot\mathbf{w})^{p+\frac{1}{\eta}}}\prod_{i=1}^{p}\left\{\frac{\gamma_i w_i}{\gamma\cdot\mathbf{w}}\right\}^{\alpha_i-1}d\mathbf{w}$$

A theoretical example

For $d = 2$, then,

$$h_\eta(w) = \eta^{-1} \frac{\gamma_1\gamma_2}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{\Gamma(\alpha_1+\alpha_2+\frac{1}{\eta})}{\{\gamma_1 w+\gamma_2(1-w)\}^{2+\frac{1}{\eta}}} \left\{\frac{\gamma_1 w}{\gamma_1 w+\gamma_2(1-w)}\right\}^{\alpha_1-1} \left\{\frac{\gamma_2(1-w)}{\gamma_1 w+\gamma_2(1-w)}\right\}^{\alpha_2-1} \tag{20}$$

$$V_\eta(x_1, x_2) = x_1^{-\frac{1}{\eta}} \left\{1 - I_{\frac{\gamma_1 x_1}{\gamma' \cdot \mathbf{x}}}(\alpha_1 + \frac{1}{\eta}, \alpha_2)\right\} + x_2^{-\frac{1}{\eta}} I_{\frac{\gamma_1 x_1}{\gamma' \cdot \mathbf{x}}}(\alpha_1, \alpha_2 + \frac{1}{\eta}) \tag{21}$$

where, $I_v(a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^v t^{a-1}(1-t)^{b-1}\, dt$ is the regularized incomplete beta function, and $a, b > 0$.

A theoretical example

Thus,

$$h^*_\eta(w) = \eta^{-1}\delta^{-1}_{\alpha_1,\alpha_2,\eta}\frac{\gamma_1\gamma_2}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\frac{\Gamma(\alpha_1+\alpha_2+\frac{1}{\eta})}{\{\gamma_1 w+\gamma_2(1-w)\}^{2+\frac{1}{\eta}}} \left\{\frac{\gamma_1 w}{\gamma_1 w+\gamma_2(1-w)}\right\}^{\alpha_1-1}\left\{\frac{\gamma_2(1-w)}{\gamma_1 w+\gamma_2(1-w)}\right\}^{\alpha_2-1} \tag{22}$$

where,

$$\delta_{\alpha_1,\alpha_2,\eta} = 1 + I_{\frac{\gamma_1}{\gamma'\cdot\mathbf{1}}}(\alpha_1+\frac{1}{\eta},\alpha_2) - I_{\frac{\gamma_1}{\gamma'\cdot\mathbf{1}}}(\alpha_1,\alpha_2+\frac{1}{\eta}) \tag{23}$$

Then,

$$\bar{F}_{X_1X_2}(x_1,x_2) = \delta^{-1}_{\alpha_1,\alpha_2,\eta}\left[x_1^{-\frac{1}{\eta}} I_{\frac{\gamma_1 x_1}{\gamma'\cdot\mathbf{x}}}(\alpha_1+\frac{1}{\eta},\alpha_2) + x_2^{-\frac{1}{\eta}}\left\{1 - I_{\frac{\gamma_1 x_1}{\gamma'\cdot\mathbf{x}}}(\alpha_1,\alpha_2+\frac{1}{\eta})\right\}\right] \tag{24}$$

A theoretical example

As $\bar{F}_{Z_1Z_2}(z_1, z_2) = \lambda \bar{F}_{X_1X_2}(\frac{z_1}{u}, \frac{z_2}{u})$, where $\lambda = \Pr(Z_1 > u, Z_2 > u)$. Then,

$$\begin{aligned}&\bar{F}_{Z_1Z_2}(z_1, z_2)\\&= \lambda u^{\frac{1}{\eta}} \delta^{-1}_{\alpha_1,\alpha_2,\eta} \left[z_1^{-\frac{1}{\eta}} I_{\frac{\gamma_1 z_1}{\gamma' \cdot \mathbf{z}}}(\alpha_1 + \tfrac{1}{\eta}, \alpha_2) + z_2^{-\frac{1}{\eta}} \left\{ 1 - I_{\frac{\gamma_1 z_1}{\gamma' \cdot \mathbf{z}}}(\alpha_1, \alpha_2 + \tfrac{1}{\eta}) \right\} \right]\end{aligned} \tag{25}$$

We further get the density of the distribution function as follows:

$$f_{Z_1Z_2}(z_1, z_2; \Theta) = \lambda u^{\frac{1}{\eta}} \delta^{-1}_{\alpha_1,\alpha_2,\eta} \frac{\gamma_1\gamma_2 \Gamma(\alpha_1 + \alpha_2 + \frac{1}{\eta})}{\eta \Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{(\gamma_1 z_1)^{\alpha_1 - 1} (\gamma_2 z_2)^{\alpha_2 - 1}}{(\gamma_1 z_1 + \gamma_2 z_2)^{\alpha_1 + \alpha_2 + \frac{1}{\eta}}} \tag{26}$$

where, parameter set $\Theta = (\alpha_1, \alpha_2, \eta, \lambda)$.

Survival distribution plots

$(\alpha_1, \alpha_2) = (2, 9), \lambda = 0.3, u = 0.5$

On-going study

- Inference
  Censored maximum likelihood estimation - Smith et al. (1997)
- Numerical examples
  - Simulation
  - Practical examples

Coles, S. G. and J. A. Tawn (1991). Modelling extreme multivariate events. *Journal of the Royal Statistical Society, Series B: Methodological 53*, 377–392.

de Haan, L. and S. I. Resnick (1977). Limit theory for multivariate sample extremes. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 40*, 317–337.

Ledford, A. W. and J. A. Tawn (1997). Modelling dependence within joint tail regions. *Journal of the Royal Statistical Society, Series B: Methodological 59*, 475–499.

Ramos, A. and A. W. Ledford (2007). A new class of models for bivariate joint tails.

Smith, R. L. (2003). Statistics of extremes, with applications in environment, insurance, and finance. *Extreme Values in Finance, Telecommunications and the Environment*, 1–78.

Smith, R. L., J. A. Tawn, and S. G. Coles (1997). Markov chain models for threshold exceedances. *Biometrika 84*, 249–268.